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Dynamics of a multipartite system undergoing matter-state–photon conversions

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Abstract

We examine the entanglement dynamics of two initially entangled qubits coupled to independent photon reservoirs and undergoing continuous matter-state–photon population transitions. We represent the decay and replenishment of matter-based bit states via photons by time-dependent generalized conversion functions. For the specific case of a sinusoidal function, we show that sudden death events in qubit–qubit entanglement anti-correlate (correlate) exactly with sudden birth events in photon–photon entanglement for the symmetric (anti-symmetric) mode of quantum conversions. We show the invariance in dynamics of all possible bipartite concurrences for various configurations of qubit–reservoir systems and highlight its crucial role in identifying a global concurrence of the multipartite system. We study the coherently driven quantum dot-cavity system as a specific application of our approach, including an analysis of evolution of its Meyer–Wallach measure with time.

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1. Introduction

The efficient conversion between photons and well-known two-level matter states such as trapped atoms [1] and excitations in nanostructures [2, 3] provides an interesting perspective to the study of entanglement dynamics in quantum information systems. In recent years, many works [4–15] have focused on the theme that a system of two initial entangled qubit states evolving under the action of the environmental variables of reservoir states experiences disentanglement in finite time or undergoes entanglement sudden death (ESD) [8–10]. These works adopt the conventional model in which there is gradual and irreversible dissipation of qubit–qubit entanglement into reservoir states. In this work, we employ a framework in which quantum information is transferred from qubit states to reservoir states and vice versa at appropriate times as is the case when photons and matter states convert from one form to another. Such a model provides a convenient approach to investigating whether net entanglement amongst various bipartite systems remains a constant of time.

The entanglement between two initial entangled qubit states is irreversibly lost with onset of the first ESD to the continuum reservoir states for qubit states undergoing purely exponential decay. There are exceptions to this rule as for instance in initially unentangled qubit systems, spontaneous emission leads to entanglement under special situations [16]. In this work, we consider a generalized qubit–reservoir interaction in which bit states can be replenished periodically at the expense of reservoir states as in the case of controlled quantum state conversions. The choice of a generalized qubit–reservoir interaction is appropriate and timely in view of significant advances in experimental techniques where spontaneous emission lifetimes in quantum dot-microcavity systems can be easily manipulated using system parameters [17]. The well-controlled emission of photons in cavity quantum electrodynamics (QED) can also be reversed in high finesse systems so that the emission process becomes almost deterministic under suitable conditions [18]. Recent experimental realization of efficient exciton–plasmon–photon conversions [2] as well as storage capabilities of photons into alternative forms of matter [3] also show potential applications of generalizing qubit–reservoir interactions in quantum systems.

In this work, we examine the entanglement dynamics of entangled qubits for some explicit choices of matter-state–photon conversion functions and for various configurations of qubit–reservoir systems. For instance, other than considering a pair of qubits which are matched with their reservoir counterparts, we also include systems in which either one reservoir or qubit is absent. The choice of a simple form (pure sinusoidal with zero decoherence) in the first instance is used to highlight important features associated with reversible conversions between photon and matter states. We also consider in detail the coherently driven quantum dot-cavity system including analysis of its Meyer–Wallach measure. The explicit relations between sums of bipartite concurrences for various configurations of quantum conversions in the multipartite qubit system as well as the influence of symmetric and antisymmetric modes of conversions on entanglement exchanges are also investigated. Lastly, we examine how the three-tangle (τ_3), which is a generalization of concurrence to three subsystems, responds to matter-state–photon conversions.

2. Entanglement in multipartite systems

To illustrate the above problem, we consider a pair of two-level qubit system with the ground (excited) state $|0\rangle_{\rm ex}$ ($|1\rangle_{\rm ex}$) corresponding to the absence (presence) of an excited state with equal creation energy at adjacent quantum dots. The quantum dots are assumed to be located far apart so that interactions between the excited states leading to the formation of multiply charged states such as charged excitons [19] are excluded and the possibility of formation of qutrit states neglected. Each qubit is coupled to its own reservoir of photons so that at $t = 0$, the reservoirs associated with different quantum dots are uncorrelated. The qubit in its higher excited state is considered to undergo quantum state conversion to photon states during which it makes a transition from $|1\rangle_{\rm ex}$ to $|0\rangle_{\rm ex}$. For simplicity, we consider that only a single photon at any allowed mode is emitted to form part of the reservoir states.

We associate the presence (absence) of matter states or photons with the existence of a two-level qubit system in the ground (excited) state so that matter states can be considered to store information. Exchange of information occurs when matter states undergo population transition to photon states and vice versa and thus information flows whenever conversions occur between matter and photon states. Hence, we use the terms 'state transfer' and 'conversion' interchangeably to describe information flow that occurs when matter states convert to photons and vice versa. We regard the processes of population transfer, information flow, quantum state transfer and conversion as unified by both local and non-local exchanges, in the spirit of an important work by Cirac *et al* [20].

We consider that the conversion process proceeds initially from a composite state of a single excited state in a quantum dot with its corresponding reservoir in the vacuum state via the simple route

$$|1\rangle_{\rm ex}\,|0\rangle_{\rm p} \longrightarrow u(t)\,|1\rangle_{\rm ex}\,|0\rangle_{\rm p} + v(t)\,|0\rangle_{\rm ex}\,|1\rangle_{\rm p}, \tag{1}$$

where $|0\rangle_{\rm p}$ denotes the reservoir state with zero photon occupation at allowed modes and $|1\rangle_{\rm p}$ denotes the presence of a single photon at any allowed mode. In this work, we reserve the term qubit only for the excited state in a quantum dot and refer to photons (denoted by index p) as reservoir states for notational convenience, bearing in mind that photons are well known [1] as 'flying qubits'. The functions $u(t)$ and $v(t)$ are considered to satisfy the relation $u(t)^2 + v(t)^2 = 1$. The characteristics of conversion between the excited state and photon states is contained in the generalized function $u(t)$ which is generally dependent on external control parameters as can be interpreted from recent investigations of quantum state conversion processes [2, 3, 17, 21]. However, the determination of the exact form for $u(t)$ is based on several parameters associated with the experimental set-up and is beyond the scope of this work. Hence, we focus on the entanglement flow associated with just simple forms for $u(t)$ and $v(t)$, assuming a global quantum system consisting of only the qubit and photon subsystem.

The reversible conversion process $|1\rangle_{\rm ex}\,|0\rangle_{\rm p} \Longleftrightarrow |0\rangle_{\rm ex}\,|1\rangle_{\rm p}$ leads to entanglement between the excited state and photon states and may last for an infinite time period for $u(t)$ that is immune to spontaneous emission processes. The excited state converts to photon states with a monotonic decrease of $u(t)$ with time while there is transfer from photon to matter states when $u(t)$ increases with time. It is common to use $u(t) \sim \exp(-t/2)$ in the presence of a large reservoir of photon states [7–9]. We now consider the joint evolution of a pair of two-level qubits undergoing conversions to photon states in uncorrelated reservoirs using a generalized initial state

$$\begin{aligned}|\Phi\rangle_0 = [&a\,|0\rangle_{\rm ex1}\,|0\rangle_{\rm ex2} + b\,|1\rangle_{\rm ex1}\,|1\rangle_{\rm ex2} + c\,|0\rangle_{\rm ex1}\,|1\rangle_{\rm ex2}\\ &+ d\,|1\rangle_{\rm ex1}\,|0\rangle_{\rm ex2}]\,|0\rangle_{\rm p1}\,|0\rangle_{\rm p2},\end{aligned} \tag{2}$$

where $i = 1, 2$ denote the two qubit–reservoir systems with associated functions $u_i(t)$. The state $|\Phi\rangle_0$ evolves as a four-qubit multipartite state influenced by the real coefficients a, b, c, d. As the conversion process is initiated only by the formation of photons from the excited state, components of $|\Phi\rangle_0$ associated with $|0\rangle_{\rm ex}$ at $t = 0$ do not evolve with time. Thus for instance, terms associated with coefficients d evolve as

$$d\,|1\rangle_{\rm ex1}\,|0\rangle_{\rm ex2}\,|0\rangle_{\rm p1}\,|0\rangle_{\rm p2} \longrightarrow d[u_1(t)\,|1\rangle_{\rm ex1}\,|0\rangle_{\rm p1} + v_1(t)\,|0\rangle_{\rm ex1}\,|1\rangle_{\rm p1}]\,|0\rangle_{\rm ex2}\,|0\rangle_{\rm p2}. \tag{3}$$

Using equations (1) and (2) and tracing out the reservoir states, we obtain a time-dependent qubit–qubit reduced density matrix in the basis $(|0\,0\rangle, |0\,1\rangle|1\,0\rangle|1\,1\rangle)$

$$\rho_{\rm ex1,ex2}(t) = \begin{pmatrix} f_1(t) & f_5(t) & f_6(t) & f_7(t)\\ f_5(t) & f_2(t) & f_8(t) & f_9(t)\\ f_6(t) & f_8(t) & f_3(t) & f_{10}(t)\\ f_7(t) & f_9(t) & f_{10}(t) & f_4(t)\end{pmatrix}, \tag{4}$$

where for $t \geqslant 0$, the matrix elements evolve as

$$\begin{aligned}f_1(t) &= a^2 + b^2 v_1(t)^2 v_2(t)^2 + c^2 v_2(t)^2 + d^2 v_1(t)^2,\\ f_2(t) &= b^2 v_1(t)^2 u_2(t)^2 + c^2 u_2(t)^2,\\ f_3(t) &= b^2 u_1(t)^2 v_2(t)^2 + d^2 u_1(t)^2,\end{aligned}$$

$$f_4(t) = b^2 u_1(t)^2 u_2(t)^2, \qquad f_5(t) = acu_2(t),$$
$$f_6(t) = adu_1(t) + bcu_1(t)v_2(t)^2,$$
$$f_7(t) = abu_1(t)u_2(t), \qquad f_8(t) = cdu_1(t)u_2(t),$$
$$f_9(t) = bcu_1(t)u_2(t)^2, \qquad f_{10}(t) = bdu_1(t)^2 u_2(t).$$

The photon–photon reduced density matrix $\rho_{\mathrm{p1,p2}}$ can similarly be obtained by tracing out the excited states and possesses a form complementary to equation (4) with $u_i \leftrightarrow v_i$. For $c = d = 0$, reduced bipartite density matrices $\rho_{\mathrm{ex1,ex2}}$, $\rho_{\mathrm{p1,p2}}$, $\rho_{\mathrm{ex1,p1}}$ and $\rho_{\mathrm{ex1,p2}}$ assume simple forms with the well-known X-state structure which preserve their X-form during evolution. The density matrix of an X-state has the following X-form:

$$\rho_{X,Y}(t) = \begin{pmatrix} p & 0 & 0 & x \\ 0 & q & y & 0 \\ 0 & y^\star & r & 0 \\ x^\star & 0 & 0 & s \end{pmatrix}, \tag{5}$$

where normalization and positivity conditions, $\mathrm{Tr}\rho_{X,Y}(t) = 1$ and $\rho_{X,Y}(t) > 0$, require that p, q, r, s are non-negative, $p+q+r+s = 1$ and off-diagonal terms x and y satisfy $|x| \leqslant \sqrt{ps}$ and $|y| \leqslant \sqrt{qr}$

In order to study the conditions under which ESD and entanglement sudden birth (ESB) occur, we evaluate the concurrence [23] for the appropriate density matrix using $\mathcal{C}(t) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}$ where λ_i are eigenvalues in decreasing order of the Hermitian matrix $\widetilde{\rho} = \rho\left(\sigma_y^1 \otimes \sigma_y^2\right)\rho^*\left(\sigma_y^1 \otimes \sigma_y^2\right)$ where σ_y belongs to the set of Pauli matrices. ρ^* denotes the complex conjugation of ρ in the standard basis (4). For density matrices associated with all possible bipartite partitions of two qubits, we obtain for $c = d = 0$

$$\begin{aligned}
\mathcal{C}_{\mathrm{ex1,ex2}}(t) &= \max\{0, 2u_1u_2(|ab| - b^2 v_1 v_2)\}, \qquad (6)\\
\mathcal{C}_{\mathrm{p1,p2}}(t) &= \max\{0, 2v_1v_2(|ab| - b^2 u_1 u_2)\},\\
\mathcal{C}_{\mathrm{ex1,p1}}(t) &= \max\{0, 2b^2 u_1 v_1\},\\
\mathcal{C}_{\mathrm{ex1,p2}}(t) &= \max\{0, 2abu_1v_2 - 2b^2 u_1u_2v_1v_2\},
\end{aligned}$$

where the forms for $\mathcal{C}_{\mathrm{ex2,p2}}(t)$ and $\mathcal{C}_{\mathrm{ex2,p1}}(t)$ can be obtained by interchanging $u_1 \leftrightarrow u_2$ and $v_1 \leftrightarrow v_2$. From equation (2), we note that $c = d = 0$ corresponds to the physical situation where the pair of quantum dot system exists simultaneously in excited or ground states. Thus, we exclude the case where the quantum dots exist in different qubit states at $t = 0$. It is important to note that due to normalization and positivity conditions, partitions $c = d = 0$ and $a = b = 0$ cannot be satisfied simultaneously if the X-form in equation (4) is to be maintained.

2.1. Symmetric mode of quantum state conversion

To simplify analysis of the entanglement dynamics of the reduced density matrix in equation (4) and associated pairwise interactions, we consider the symmetric mode of conversion where the information flow from both qubits to their reservoir counterparts occurs in phase. We consider the simple case where $u_1 = u_2 = \frac{1}{\sqrt{2}}\sqrt{1+p}$ and $v_1 = v_2 = \frac{1}{\sqrt{2}}\sqrt{1-p}$. Using equation (6), we obtain an explicit relation involving all possible bipartite concurrences in the region $\max\left(-1, 1 - 2\left|\frac{a}{b}\right|\right) < p < \min\left(1, 2\left|\frac{a}{b}\right| - 1\right)$ and $\frac{a}{b} > \frac{1}{2}$:

$$\begin{aligned}
\mathcal{C}_{\mathrm{g}}(t) &= \mathcal{C}_{\mathrm{ex1,ex2}}(t) + \mathcal{C}_{\mathrm{p1,p2}}(t) + \left|\frac{a}{b}\right| [\mathcal{C}_{\mathrm{ex1,p1}}(t) + \mathcal{C}_{\mathrm{ex2,p2}}(t)] - \mathcal{C}_{\mathrm{ex1,p2}}(t) - \mathcal{C}_{\mathrm{ex2,p1}}(t)\\
&= 2|ab|.
\end{aligned} \tag{7}$$

We define $\mathcal{C}_{\mathrm{g}}(t)$ as the global concurrence composed of all possible bipartite concurrences with appropriate weighting parameters and dependent only on the initial configuration of the multipartite system. $\mathcal{C}_{\rm g}(t)$ remains unchanged with time and the overall distribution of entanglement among various subsystems involved in pairwise interactions is conserved in the absence of decoherence or dissipative mechanisms [24, 25]. The global concurrence $\mathcal{C}_{\rm g}(t)$ therefore possesses the important property of invariance in dynamics of all possible bipartite concurrences. An increase in entanglement between any two parties would be at the expense of information that is shared between other parties and is dependent on the initial entangled state amplitude parameters a and b. In figure 1, we show the regions of invariance of dynamics of a multipartite system (shaded) undergoing the symmetric mode of quantum state conversion as considered in equation (7). The invariance in dynamics for the specific configuration involving matched pairs of the qubit–reservoir system and employing the Jaynes–Cummings model operating in the symmetric mode has also been shown in an earlier work by Yönaç *et al* [10]. However, the crucial role played by the property of invariance in defining the global invariance was unexplored in Yönaç's work. It is important to note that in the definition for $\mathcal{C}_{\rm g}(t)$, the inter qubit–photon partitions (ex1, p2) and (ex2, p1) are subtracted from the superpositions of the other bipartite partitions. The possibility of determining a global concurrence that is an invariant of dynamics of all possible bipartite concurrences for the more general N pair of the qubit–reservoir system will be pursued in a future work.

Substituting $p = \cos(\omega t)$, we can verify that multiple ESD and ESB events occur at periodic intervals:

$$\begin{aligned} t_{\rm ex1,ex2} &= \frac{1}{\omega}\left[\cos^{-1}\left(1-2\left|\frac{a}{b}\right|\right)+2n\pi\right], \\ t_{\rm p1,p2} &= \frac{1}{\omega}\left[\cos^{-1}\left(2\left|\frac{a}{b}\right|-1\right)+2n\pi\right], \end{aligned} \tag{8}$$

where increasing values of integer n signify successive ESB and ESD events in qubit–qubit and photon–photon entanglement. At $b = 2a, t_{\rm ex1,ex2} = t_{\rm p1,p2}$ so that times of death events associated with the bipartite partition matrix $\rho_{\rm ex1,ex2}$ coincide with birth events for its counterpart partition matrix $\rho_{\rm p1,p2}$. The maximum concurrence of $\mathcal{C} = 2|ab|$ oscillates between these two partitions at periodic intervals of $2\pi/\omega$. Such a pattern of information transfer is intuitively expected as the information flows from both qubit system to their reservoir counterpart occur in tandem. The situation is reversed in the case of the antisymmetric mode of quantum state conversion as will be shown in the following section. While the explicit choice of a sinusoidal function for $p = \cos(\omega t)$ has not been derived from an appropriate model Hamiltonian, equation (8) captures some salient features associated with reversible mapping of a photon state to and from the ground state of quantum dot systems. It is to be noted that similar trends in birth and death events have been obtained for $u(t) \sim \exp(-t/2)$ in an earlier work by López and coworkers [7].

It is to be noted that for $a = b = 0$, reduced bipartite density matrices $\rho_{\rm ex1,ex2}$ and $\rho_{\rm p1,p2}$ also assume simple forms with the well-known X-state structure, the only difference from the $c = d = 0$ case being that the terms $f_4(t) = 0$ and $f_7(t) = 0$. We also note similarities in the role played by functions $f_7(t) = 0$ and $f_8(t) = 0$. Accordingly, the various bipartite concurrences can be easily calculated and analysis as in equation (6) can be made. Results similar to the $c = d = 0$ case are expected to be obtained and we will therefore not pursue investigation of the $a = b = 0$ situation any further.

To illustrate a symmetric mode of quantum state conversion with non-zero values for c, we have shown in figure 2(a) numerical results of the evolution of concurrence between the two qubits $\mathcal{C}_{\rm ex1,ex2}(t)$ (solid line) and two photons $\mathcal{C}_{\rm p1,p2}(t)$ (short dashed line) for the initial state of equation (2). We have used $a = \sqrt{1/3}, b = \sqrt{1.9/3}$ and $c = \sqrt{0.1/3}$ with $u_1 = v_2 = \sqrt{1/2(1+\cos(5\lambda t)}\exp(-0.25\lambda t)$. The form of $u(t)$ (long dashed lines in figure 2(b)) describes approximately the beating phenomena with weak damping that occurs

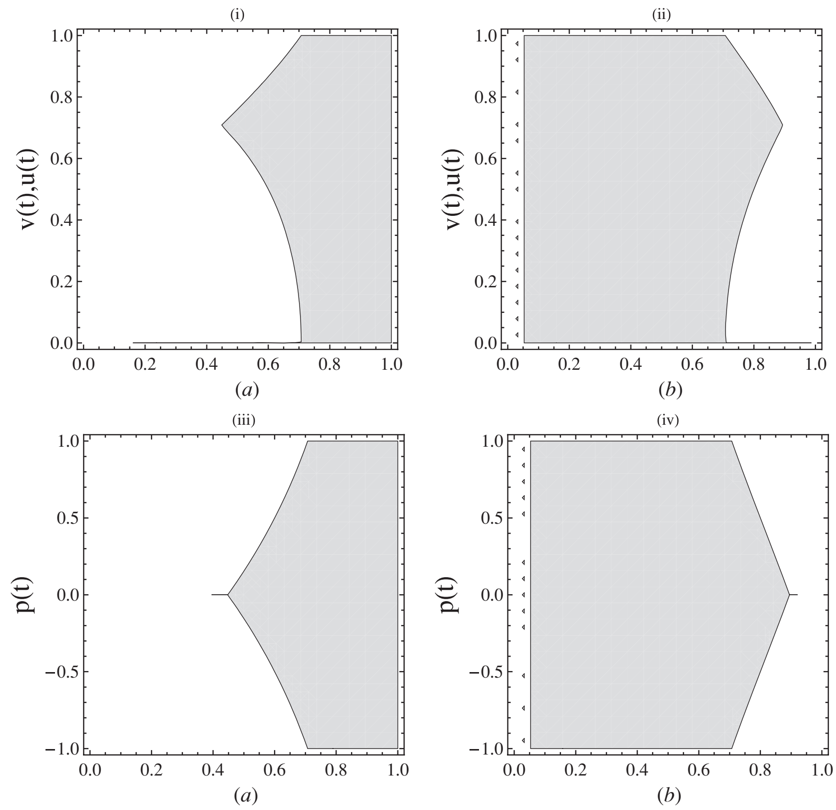

Figure 1. (i)–(iv) Shaded regions represent entanglement dynamics of the multipartite system in which $\mathcal{C}_{\mathrm{g}}(t)$ in equation (7) remains invariant. Panels (i) and (ii) correspond to $u(t)$ or $v(t)$ as y-axis variables while (iii) and (iv) correspond to $p(t)$ as the y-axis variable. In (i) and (ii), both $u(t)$ and $v(t)$ as y-axis variables yield the same invariance regions due to inherent symmetry in equation (6). a and b denote the initial entangled state amplitude parameters.

between the isolated molecule-nanoparticle system undergoing conversions via coherent energy transfer [22]. We have assumed that photons take on an ancillary role during the energy transfer. The concurrence in the qubit–qubit subsystem is transferred to the bipartite photon system depending on the characteristics of quantum state exchanges determined by $u(t)$. Figure 2(b) shows that at $\lambda t \geqslant 2.3$, evolution of $\mathcal{C}_{\mathrm{ex1,p1}}(t)$ (solid line) matches that of $u(t)$. The fine interplay in entanglement exchanges between the intra qubit–photon concurrence $\mathcal{C}_{\mathrm{ex1,p1}}(t)$ and inter qubit–photon concurrence $\mathcal{C}_{\mathrm{ex1,p2}}(t)$ is dependent on the initial state parameters, a, b, c.

2.2. *Antisymmetric mode of quantum state conversion*

In the case of the antisymmetric mode of quantum state conversion, the information flow from qubit to reservoir in one system is compensated by information flow from reservoir to qubit in the adjacent subsystem for which we express $u_1 = v_2 = u$, $v_1 = u_2 = v$. This means that the system of two qubit–reservoir states undergoes matter-state–photon conversions such that formation of photons in one qubit–reservoir is correlated with the formation of excited states in the adjacent qubit–reservoir system. We can expect exact matching in qubit–qubit

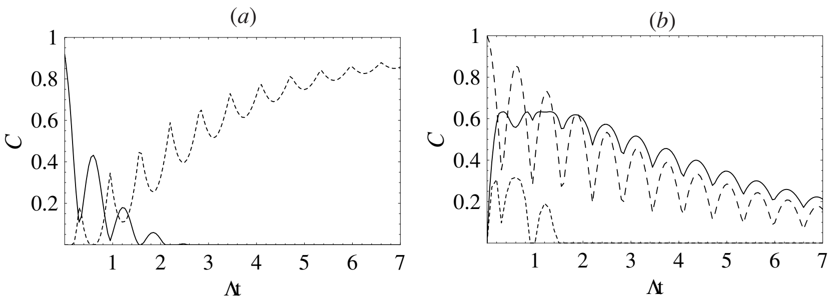

Figure 2. (*a*) Evolution of two-qubit concurrence $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ (solid line) and $\mathcal{C}_{\mathrm{p1,p2}}(t)$ (short dashed), for the initial state of equation (2) with $a = \sqrt{1/3}, b = \sqrt{1.9/3}$ and $c = \sqrt{0.1/3}$. (*b*) Evolution of $\mathcal{C}_{\mathrm{ex1,p1}}(t)$ (solid line) and $\mathcal{C}_{\mathrm{ex1,p2}}(t)$ (short dashed), for the same initial state as in figure 4(*a*). $u_1 = u_2 = \sqrt{1/2(1 + \cos(5\lambda t)}\exp(-0.25\lambda t)$ is shown in long dashed lines.

and reservoir–reservoir concurrences under these conditions. Using $u_1 = \frac{1}{\sqrt{2}}\sqrt{1+p}$ and $u_2 = \frac{1}{\sqrt{2}}\sqrt{1-p}$ and equation (6), we obtain an explicit expression for the global concurrence

$$\begin{aligned}\mathcal{C}_{\mathrm{g}}(t) &= \mathcal{C}_{\mathrm{ex1,p2}}(t) + \mathcal{C}_{\mathrm{ex2,p1}}(t) + \left|\frac{a}{b}\right| [\mathcal{C}_{\mathrm{ex1,p1}}(t) + \mathcal{C}_{\mathrm{ex2,p2}}(t)] - \mathcal{C}_{\mathrm{ex1,ex2}}(t) - \mathcal{C}_{\mathrm{p1,p2}}(t)\\ &= 2|ab|. \end{aligned} \tag{9}$$

Equation (9) becomes identical to equation (7) when the notational change $\mathrm{ex}_2 \leftrightarrow \mathrm{p}_2$ is made as is the case when association of the excitation state $|1\rangle_{\mathrm{ex}}$ with the generalized function u_i is made. As mentioned earlier, the evolution of qubit–qubit entanglement matches that of reservoir–reservoir entanglement and the two bipartite subsystems ($\rho_{\mathrm{ex1,ex2}}$ and $\rho_{\mathrm{p1,p2}}$) undergo birth and death events concurrently. For $p = \cos(\omega t)$, we obtain

$$t_{\mathrm{ex1,ex2}} = t_{\mathrm{p1,p2}} = \frac{1}{\omega}\left[\sin^{-1}\left(2\left|\frac{a}{b}\right|\right) + 2n\pi\right]. \tag{10}$$

At $t = 0, u_1(0) = 1, u_2(0) = 0$ and we obtain $\mathcal{C}_{\mathrm{ex1,ex2}} = \mathcal{C}_{\mathrm{p1,p2}} = \mathcal{C}_{\mathrm{ex1,p1}} = \mathcal{C}_{\mathrm{ex2,p2}} = \mathcal{C}_{\mathrm{ex2,p1}} = 0$. At $t = 0$, all quantum information is carried by the inter qubit–reservoir bipartite partition, $\mathcal{C}_{\mathrm{ex1,p2}}(0) = 2|ab|$. Figure 3 shows numerical results of the evolution of concurrence between the two qubits (solid line) and two photons (dotted line) for the initial state of equation (2) with $a = \sqrt{0.9/3}, b = \sqrt{1.5/3}, c = \sqrt{0.2/3}$ and $d = \sqrt{0.4/3}$ for $u_1 = v_2 = \sqrt{1/2(1 + \cos(2\lambda t)}\exp(-0.1\lambda t)$. The higher entanglement contained initially in the photon–photon subsystem is partially transferred to the bipartite qubit system based on the quantum state exchanges determined by $u(t)$. The slight discrepancy in the concurrence of the bipartite partition (ex1, ex2) and (p1, p2) can be attributed to non-zero values for the initial state amplitudes c and d in equation (2).

2.3. *Absent qubit or reservoir*

Here we briefly analyze entanglement flow characteristics in multipartite system configurations in which either a qubit or a reservoir is absent. We aim to examine the effect of changes in the configurations of qubit–reservoir systems on the invariance of non-zero pairwise concurrences. When $u_1 = u, u_2 = 1, v_1 = v$ and $v_2 = 0$, the second reservoir can be considered absent. Accordingly, all pairwise concurrences involving a non-existent second reservoir are zero, $\mathcal{C}_{\mathrm{p1,p2}}(t) = \mathcal{C}_{\mathrm{ex1,p2}}(t) = \mathcal{C}_{\mathrm{ex2,p2}}(t) = 0$. The second qubit is not coupled to any reservoir states at $t = 0$; however, this changes with time as $\mathcal{C}_{\mathrm{ex2,p1}}(t) = 2|ab|v$ implying that at $t > 0$, the qubit

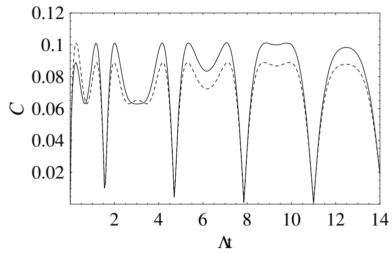

Figure 3. Evolution of two-qubit concurrence $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ (solid line) and qubit–photon concurrence $\mathcal{C}_{\mathrm{p1,p2}}(t)$ (short dashed) for the initial state of equation (2) with $a = \sqrt{0.9/3}$, $b = \sqrt{1.5/3}$, $c = \sqrt{0.2/3}$ and $d = \sqrt{0.4/3}$ and $u_1 = v_2 = \sqrt{1/2(1+\cos(2\lambda t)}\exp(-0.1\lambda t)$

adopts the reservoir states belonging to the first qubit. We can easily show using equation (6) that $\mathcal{C}^2_{\mathrm{ex2,p1}} + \mathcal{C}^2_{\mathrm{ex1,ex2}} = 4a^2b^2$ which implies conservation of concurrence associated with the second qubit, a scenario which one can expect intuitively. We also obtain an explicit expression for the global concurrence based on the invariance of the remaining non-zero pairwise concurrences:

$$\mathcal{C}_{\mathrm{g}}(t) = \frac{\mathcal{C}_{\mathrm{ex1,ex2}} \times \mathcal{C}_{\mathrm{ex2,p1}}}{\mathcal{C}_{\mathrm{ex1,p1}}} = 2a^2. \tag{11}$$

The entanglement between the two qubit states is compromised by the strengthening of inter qubit–photon concurrence, $\mathcal{C}_{\mathrm{ex2,p1}}(t)$. Thus, a system of two qubits where only one reservoir is present evolves such that the sole reservoir becomes the source of conversion or decoherence over time. In the related case where the second qubit is absent, we set $u_1 = u$, $u_2 = 0$, $v_1 = v$ and $v_2 = 1$; the system is composed of one qubit and two reservoirs. It can be easily shown that the sole qubit adopts the unattached reservoir as a source of quantum state conversion or decoherence over time. By making the association, $\mathrm{ex}_2 \leftrightarrow \mathrm{p}_2$, we obtain the following invariance relations from equation (11):

$$\begin{aligned} &\mathcal{C}^2_{\mathrm{p2,p1}} + \mathcal{C}^2_{\mathrm{ex1,p2}} = 4a^2b^2 \\ &\mathcal{C}_{\mathrm{g}}(t) = \frac{\mathcal{C}_{\mathrm{ex1,p2}} \times \mathcal{C}_{\mathrm{p2,p1}}}{\mathcal{C}_{\mathrm{ex1,p1}}} = 2a^2. \end{aligned} \tag{12}$$

It is important to note that results obtained so far are generic to any two-level qubit system undergoing quantum state conversion and therefore not specific to excitations in the quantum dot system considered here.

3. The Meyer–Wallach measure

Here we analyze the change in the Meyer–Wallach measure (MW) with generalized function u_i for the pair of qubit–reservoir system in equation (2). We examine how a different measure of entanglement influences information flow when matter states convert to photons and vice versa. The MW measure is a monotone measure defined [26] as a single scalar measure of pure state entanglement for the three- and four-qubit cases:

$$Q = \frac{1}{n}\sum_{k=1}^{n} 2\left(1 - \mathrm{Tr}\left[\rho_k^2\right]\right), \tag{13}$$

where ρ_k is the reduced density matrix of the kth qubit which is obtained after tracing out all the remaining qubits. The MW measure has drawbacks in that it is unable to distinguish states which are fully inseparable from states which are separable into states of some set of subsystems. Nevertheless we will, for simplicity reasons, use it in our work as a crude quantity to analyze results obtained using concurrence, a purely bipartite measure.

For $c = d = 0$, the simple forms for the one-qubit reduced density matrices allow us to obtain an explicit expression for the MW measure:

$$Q = \frac{1}{2}\left[\sum_{i=1}^{2}\left(1-\left(a^2+b^2u_i^2\right)^2-b^4v_i^2\right)+\left(1-\left(a^2+b^2v_i^2\right)^2-b^4u_i^2\right)\right]. \tag{14}$$

Maximum $Q = 1$ is obtained for $a = 0, b = 1$ with $u_1 = u_2 = \frac{1}{\sqrt{2}}$ for the following state:

$$|\Phi\rangle_m = \frac{1}{2}[|11\rangle_{\rm ex}\,|00\rangle_{\rm p} + |00\rangle_{\rm ex}\,|11\rangle_{\rm p} + |01\rangle_{\rm ex}\,|10\rangle_{\rm p} + |10\rangle_{\rm ex}\,|01\rangle_{\rm p}],$$

which involves all possible combinations of the quantum state where either the qubit or its corresponding reservoir is occupied but not both at the same time. For $a = b = \frac{1}{\sqrt{2}}$, we obtain $Q = \frac{1}{2}\left(1+u_1^2-u_1^4+u_2^2-u_2^4\right)$, and a constant value of $Q = 0.5$ is obtained for $(u_1 = 1, u_2 = 0)$, $(u_1 = 0, u_2 = 1)$, $(u_1 = 1, u_2 = 1)$ or $(u_1 = 0, u_2 = 0)$ partitions. In the following section, we include an analysis of the evolution of the MW measure in the coherently driven quantum dot-cavity system.

4. Entanglement of the coherently driven quantum dot-cavity system

In this section, we consider two noninteracting quantum dot excitonic qubits which are placed inside high-Q single mode cavities and which interact individually with a single mode of the radiation field. We also consider each excitonic system to be coherently driven by an external quantized field of a different mode. This problem is analogous to Wilken and Meystre's modified Jaynes–Cummings model [27] in which two different modes are supported by a resident atom in a cavity. The exciton–field interaction is thus given by the Hamiltonian [27] $(\hbar = 1)$

$$H = g[\sigma_+(a+b) + \sigma_-(a^\dagger + b^\dagger)], \tag{15}$$

where g is the exciton-cavity photon coupling constant which is assumed to be the same as the exciton-external field coupling constant and a and b are annihilation operators for the two modes interacting with the excitonic system. σ_+ and σ_- are the respective Pauli's raising and lowering matrix operators. We assume for simplicity that phonon interactions are absent bearing in mind that inclusion of lattice vibrations will not affect the final outcome of our result. We define normal-mode operators

$$A = \frac{1}{\sqrt{2}}(a+b), \qquad B = \frac{1}{\sqrt{2}}(a-b),$$

which obey the Bose commutation relations, $[A, A^\dagger] = [B, B^\dagger] = 1$. These commutation relations simplify equation (15) to the following form [27]:

$$H = \sqrt{2}g[\sigma_+A + A^\dagger\sigma_-], \tag{16}$$

which is noticeably independent of the antisymmetric combination operator B. Following [28], we represent photons in the external field as well as cavity field in the form of coherent states:

$$|\alpha\rangle_a\,|\beta\rangle_b = \left|\frac{1}{\sqrt{2}}(\alpha+\beta)\right\rangle_A \left|\frac{1}{\sqrt{2}}(\alpha-\beta)\right\rangle_B.$$

Finally, we substitute Wilken and Meystre's simple expression for the probability of finding the atom in the excited state [27] for $u(t)^2$ (see equation (1)) which yields the survival probability of the exciton in a quantum dot at $t > 0$:

$$\begin{aligned} u^2(t,\alpha,\beta,g) &= \left| \cos(\sqrt{2}gt\sqrt{K^\dagger K}) \left| \frac{1}{\sqrt{2}}(\alpha+\beta) \right\rangle_A \right|^2 \\ &= \frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{-\frac{|\alpha+\beta|^2}{2}} \sum_{n=0}^{\infty} \left(\frac{|\alpha+\beta|^2}{2} \right)^n \frac{1}{n!} \cos(2\sqrt{2}gt\sqrt{n+1}), \end{aligned} \tag{17}$$

where $K = A^\dagger A$ is the number operator for the symmetric normal mode. Accordingly, we identify $v(t)^2$ with the survival probability of the coherent state of photons in the symmetric mode, $\left|\frac{1}{\sqrt{2}}(\alpha+\beta)\right\rangle_A$. We thus arrive at two entities which exchange quantum information, the exciton and the coherent state of photons in the symmetric combination mode. A similar expression as in equation (17) has been obtained for the coherently driven quantum dot-cavity system with phonon interactions in [29]. Klimov and Chumakov [30] and Azuma [31] have simplified evaluation of the infinite series in equation (17) by converting the series into a sum of two integrals. It was shown [30, 31] that the first integral describes the initial collapse of the excited system in the semi-classical limit while the second integral incorporating quasi-chaotic dynamics occurs due to quantum correction.

We now consider the joint evolution of a pair of noninteracting coherently driven quantum dot-cavity system interacting with coherent photon states in the symmetric mode. To illustrate the effect of the parameters α, β and the exciton-cavity photon coupling constant g on the entanglement dynamics, we show in figure 4(*a*) numerical results of concurrence describing the entanglement of the pair of quantum dot excitons, $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ in the initial state of equation (2) with $a = \sqrt{1/2}, b = \sqrt{1/2}, c = d = 0$. The solid line corresponds to $u_1 = u_2 = u(t, \alpha = 3.8, \beta = 2, g = 1.45)$ substituted in $u(t)^2$ (see equation (17)). The dashed line applies to $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ evaluated using $u_1 = u(t, 3.8, 2, 1.45)$ and $u_2 = u(t, 3.8, 2, 0.35)$. The figure shows the typical collapses and revivals [27] present in the concurrence, $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$. The frequency of these collapses and revivals increases with the coupling constant g. One can detect a region of small amplitude of oscillations for large g (solid line) which is disrupted by sizable amplitudes when two quantum dot-cavity systems with one having a smaller value of g are considered (dotted line in figure 4(*a*)). The entanglement of two unequal subsystems, one of larger g and another of smaller g, leads to a decrease in frequency of revivals as well as reduction in amplitude of revivals at $10 \leqslant t \leqslant 14$. We note similar trends in figures 4(*b*) and 4(*c*) where we evaluate the Meyer–Wallach measure Q for $0 \leqslant t \leqslant 10$ and $10 \leqslant t \leqslant 15$ respectively. The dashed line corresponds to Q evaluated using $u_1 = u_2 = u(t, \alpha = 3.8, \beta = 2, g = 1.45)$ while the solid line corresponds to Q evaluated using $u_1 = u(t, 3.8, 2, 1.45)$ and $u_2 = u(t, 3.8, 2, 0.35)$. We note that Q is bounded by the upper limit of $Q = 3/4$ which is obtained when $u_1 = u_2 = a = b = \sqrt{1/2}$.

In figure 4(*d*), we highlight the anti-correlation between qubit–qubit and photon–photon ($\mathcal{C}_{\mathrm{p1,p2}}(t)$) entanglements in the symmetric mode where $u_1 = u_2 = u(t, 2, 2, 1)$. We note that increasing the value of g with respect to α and β yields a region ($2.5 \leqslant t \leqslant 7$) where the oscillations reach almost zero amplitude after the initial collapse of the exciton. A quasi-chaotic region ($10 \leqslant t \leqslant 25$) in which frequent exchanges between the exciton–exciton subsystem and the bipartite photon system is obtained where the frequency of exchanges is influenced by the initial state parameters, a and b.

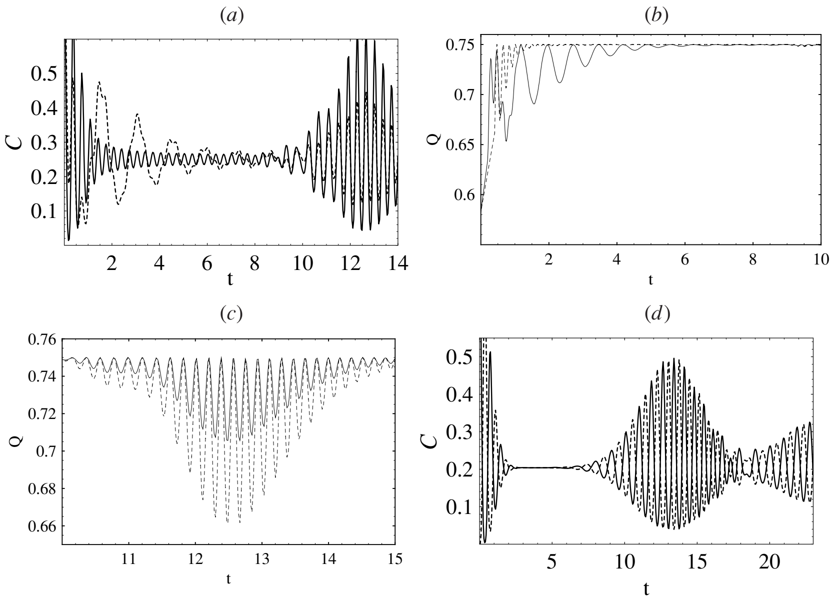

Figure 4. (*a*) Concurrence $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ of a pair of coherently driven quantum dot-cavity systems with the initial state in equation (2) with $a = \sqrt{1/2}, b = \sqrt{1/2}, c = d = 0$ and $u_1 = u_2$ with $\alpha = 3.8, \beta = 2, g = 1.45$ (see equation (17)) (solid line). The dashed line applies to $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ evaluated using $u_1 = u(t, 3.8, 2, 1.45)$ and $u_2 = u(t, 3.8, 2, 0.35)$. Time t is taken to be a dimensionless quantity. (*b*) and (*c*) Evolution of the Meyer–Wallach measure Q at $0 \leqslant t \leqslant 10$ and $10 \leqslant t \leqslant 15$ in which the dashed line corresponds to Q evaluated using $u_1 = u_2 = u(t, 3.8, 2, g = 1.45)$ while the solid line corresponds to Q evaluated using $u_1 = u(t, 3.8, 2, 1.45)$ and $u_2 = u(t, 3.8, 2, 0.35)$. (*d*) Evolution of $\mathcal{C}_{\mathrm{ex1,ex1}}(t)$ (solid line) and $\mathcal{C}_{\mathrm{p1,p2}}(t)$ (dashed line), with the initial state in equation (2) with $a = \sqrt{1.5/3}, b = \sqrt{1.2/3}, c = \sqrt{0.2/3}$ and $d = \sqrt{0.1/3}, u_1 = u_2 = u(t, 2, 2, 1)$.

5. Three-tangle dynamics

In this section, we briefly examine how the purely tripartite entanglement or three-tangle (τ_3), an important measure of entanglement in three-qubit states [32], responds to matter-state–photon conversions. We consider a simple system of two qubits interacting with a sole reservoir with the initial state

$$|\Psi\rangle_0 = a\,|0\rangle_{\mathrm{ex1}}\,|0\rangle_{\mathrm{ex2}}\,|0\rangle_{\mathrm{p}} + b\,|1\rangle_{\mathrm{ex1}}\,|1\rangle_{\mathrm{ex2}}\,|0\rangle_{\mathrm{p}}\,, \tag{18}$$

which we rewrite as a state in a three-qubit Hilbert space, $|\Psi\rangle_0 = a\,|000\rangle + b\,|110\rangle$. We simplify the analysis by considering that photons formed in the reservoir states are due to a single qubit and no allowed reservoir mode can be occupied by photons from both qubit sources at the same time. This requirement ensures a tractable conversion rule

$$|110\rangle \longrightarrow u\,|110\rangle + v\,|101\rangle + w\,|011\rangle\,, \tag{19}$$

where $u^2 + v^2 + w^2 = 1$. Using equations (18) and (19), we obtain an expression for $|\Psi\rangle_t$ which describes the time evolution of $|\Psi\rangle_0$. We evaluate the three-tangle τ_3 using its definition as the modulus of Cayley's hyperdeterminants [33], $\tau_3 = 4|d_1 - 2d_2 + 4d_3|$, where d_i can be expressed in terms of the coefficients (ψ_{ijk}) of $|\Psi\rangle_t = \sum \psi_{ijk}\,|ijk\rangle$ as described in [32]. Using equation (19)

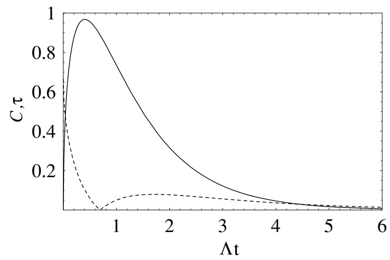

Figure 5. Evolution of tripartite entanglement (τ_3) (solid line) and two-qubit concurrence $\mathcal{C}_{\mathrm{ex1,ex2}}(t)$ (dotted line) for the initial state of equation (18) with $a = \sqrt{1/3}$ and $u = v = \sqrt{1/2}\exp(-\lambda t/2)$ (equation (19))

and the initial state in equation (18), we obtain $\psi_{000} = a$, $\psi_{110} = bu$, $\psi_{011} = bw$, $\psi_{101} = bv$ and expressions for d_i [32], $d_1 = d_2 = 0$, $d_3 = ab^3uvw$ and tangle $\tau_3 = 16ab^3uvw$. For $u = v$, we obtain by tracing out qubit or reservoir states, $\mathcal{C}^2_{\mathrm{ex1,ex2}} = \mathcal{C}^2_{\mathrm{ex1,p}} = 4(b^2uw - abu)^2$, and verification of the relation $\mathcal{C}^2_{\mathrm{ex1(p,ex2)}} = \mathcal{C}^2_{\mathrm{ex1,ex2}} + \mathcal{C}^2_{\mathrm{ex1,p}} + \tau_3$ [32] where $\mathcal{C}_{\mathrm{ex1(p,ex2)}}$ is the concurrence due to the entanglement of the first qubit (ex1) with the remaining excitonic qubit and photon state.

For $u_1 = u_2 = \sqrt{1/2}\exp(-\lambda t/2)$ and $a = \sqrt{1/3}$, τ_3 reaches its maximum value of $(16/3\sqrt{3})ab^3 \approx 0.97$ at $t_m = \frac{1}{\lambda}\ln\frac{3}{2}$. The stark difference between τ_3 and $\mathcal{C}_{\mathrm{ex1,ex2}}$ is highlighted in figure 5 which shows the sudden death and revival event experienced in the entanglement between the two qubits (dotted line) at $t_{\mathrm{ESD}} = t_{\mathrm{ESB}} = -\frac{1}{\lambda}\ln(1 - (\frac{a}{b})^2) \approx 0.69/\lambda$. Such salient features of bipartite exchanges are not apparent in the evolution of τ_3.

6. Conclusion

In conclusion, we have analyzed entanglement exchanges amongst various bipartite partitions which constitute a multipartite system undergoing quantum state conversions. We have analytically demonstrated that the number and timing of ESD and ESB events are dependent on the characteristics of quantum state exchanges in separate qubit–reservoir systems. We have obtained explicit relations between bipartite concurrences implying overall conservation of quantum information with superposition between some bipartite partitions giving rise to entanglement between other partitions. We have shown that the invariance in dynamics of all possible bipartite concurrences for various configurations of qubit–reservoir systems plays a crucial role in identifying a global concurrence of the multipartite system. The possibility of extending these results to the more general case of $N > 3$ pairs of qubit–reservoir systems will be explored in future works.

We have also shown that salient features of bipartite entanglements may not be revealed in tripartite entanglement of three-qubit states. Our work may have implications in the construction of network protocols of the recently proposed quantum internet [34] which employs photons that undergo reversible conversions to matter states.

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